Journal of Algebra **225,** 359–380 (2000)
doi:10.1006/jabr.1999.8131, available online at http://www.idealibrary.com on IDEAL®

Non-Classical Trigonal Curves

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Communicated by Walter Feit

Received January 5, 1999

We study trigonal Gorenstein curves of arithmetic genus $g \geq 5$ such that g_3^1 has a base point, through realizing them as canonical curves lying over a cone. Using an explicit description of such curves we compute the dimension of their moduli spaces. We also investigate non-classical trigonal Gorenstein curves; in fact, we give a complete classification of such curves for arithmetic genus $g = 5$. More generally, we classify the non-classical curves when the characteristic of the constant field is $g-1$, $g-2$, or $2g-3$. In characteristic 2 we also solve the case $g = 2^n + 1$. © 2000 Academic Press

1. INTRODUCTION

Let k be an algebraically closed field. When the characteristic of k is equal to zero, it is well known that for each point of a smooth curve of genus $g \geq 1$ there exists a sequence of exactly g integers which is the set of orders of the differentials at this point. Moreover, this set of integers is $\{0, 1, \ldots, g-1\}$, except for a finite number of points, which are called Weierstrass points.

In prime characteristic, there are examples of curves where for every point one can find a differential vanishing at this point with order bigger than $g-1$. On the other hand, Schmidt showed in [Sc] that, even in prime characteristic, there exists a sequence of integers $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{g-1})$, such that the order of the regular differentials at each point is ε, except for a finite number of points also called Weierstrass points.

Let us explain this in a more general framework, and also revise some concepts that we will need. Let C be an irreducible non-singular curve. Let ψ: $C \to \mathbb{P}^n(k)$ be a morphism such that $\psi(C)$ is not contained in a

0021-8693/00 $35.00

hyperplane and let $\mathcal{D} = \{\psi^{-1}(H) \mid H$ is a hyperplane of $\mathbb{P}^n(k)\}$ be the associated linear system of the hyperplane sections. We will consider $\psi(C)$ as a parameterized curve in $\mathbb{P}^n(k)$, and the points of C will be viewed as its branches. Given $P \in C$, an integer j is called a hermitian $\mathcal{D}$-invariant of P if there is a hyperplane intersecting the branch P with multiplicity j. There are $n+1$ $\mathcal{D}$-invariants $j_0 < \cdots < j_n$ and $j_n \leq d$, where d is the degree of $\mathcal{D}$. Moreover, there is a sequence of $\mathcal{D}$-orders $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{g-1})$ that is the sequence of hermitian $\mathcal{D}$-invariants for almost all $P \in C$. The finitely many points where $(j_0, j_1, \ldots, j_n) \neq \varepsilon$ are called $\mathcal{D}$-Weierstrass points. A linear system is called non-classical if $\varepsilon \neq (0, 1, 2, \ldots, n)$. When $\mathcal{D}$ is the canonical linear system of C, all these concepts coincide with the definitions of Schmidt [Sc].

Everything we said so far is restricted to the non-singular case. In this work we study trigonal Gorenstein curves that are singular, so we need an extension of the previous theory. The approach we adopt here applies to all Gorenstein curves, and goes along the following lines.

Let C be an irreducible curve of arithmetic genus g, and $\tilde{C}$ its non-singular model. We assume that C is non-hyperelliptic and that it is a Gorenstein curve, in the sense that all its local rings are Gorenstein. This means that the morphism $\tilde{C} \to \mathbb{P}^{g-1}(k)$ induces an isomorphism of C onto a curve in $\mathbb{P}^{g-1}(k)$. We identify C with this projective curve, called the canonical curve, whose degree is $2g-2$ (cf. [H] or [Ro]). Thus we study the linear system of $\tilde{C}$ associated to the canonical morphism $\tilde{C} \to C \subset \mathbb{P}^{g-1}(k)$ as before. Finally, a Gorenstein curve is said to be non-classical if the linear system associated to the canonical morphism of C is non-classical.

Schmidt gave examples of non-classical curves of genus 3 and 4 (cf. [Sc, Sect. 6]). The curves of genus smaller than 3, and more generally, in characteristic different from two, the hyperelliptic curves are always classical. In the non-singular case, all non-classical curves of genus 3 and 4 have been classified by Komiya (cf. [Ko]). Freitas and Stöhr (cf. [F-S]) extended the classification to non-classical Gorenstein curves of arithmetic genus 3 and 4.

A canonical (consequently, Gorenstein) curve of arithmetic genus 4 lies on a quadratic surface and admits a linear system of dimension 1 and degree 3; that is, the curve is trigonal. So, in extending these classification results to higher genus, it is natural to start by studying non-classical curves that are trigonal and Gorenstein. Here we will study the case where the g_3^1 has a base point.

In [R-S], it is proved that a trigonal Gorenstein curve of arithmetic genus $g \geq 5$ carries a g_3^1 with base point, if and only if the Maroni invariant of the trigonal curve is equal to zero. Moreover, such a curve is isomorphic to a canonical curve lying over a cone in $\mathbb{P}^{g-1}(k)$. Therefore, by local parameterizations, it is possible to describe the curve by explicit equations.

Here, using these equations, we compute the dimension of the moduli spaces of trigonal Gorenstein curves of arithmetic genus g with Maroni invariant equal to zero.

We also classify the non-classical trigonal Gorenstein curves of arithmetic genus 5 when the g_3^1 has a base point. This follows from some general classification results that are valid for any arithmetic genus $g \geq 5$, that we show here. More precisely, denoting by p the characteristic of the constant field, we classify all non-classical trigonal Gorenstein curves such that the linear system has a base point when $g = (p+3)/2$ or $g = p+2$ and, in characteristic 2, we also solve the case $g = 2^n + 1$. These results cover all the possibilities for $g = 5$, except for the case $p = 5$, that we treat directly.

This last case illustrates well the difficulties involved in extending this classification to higher genus, although use of some general ideas (such as the Cartier operator) allows us to cut the calculations down to reasonable proportions.

It is worth pointing out that, as a by-product of our classification results, for every value $p \geq 2$ of the constant field characteristic there exists some non-classical trigonal curve.

These results were announced in [R].

The author thanks Professor Karl-Otto Stöhr for many fruitful discussions.

2. TRIGONAL GORENSTEIN CURVES

The approach of this paper proceeds from the study of trigonal curves that was carried out in [R-S]. So, in this section, we summarize some conclusions from that paper that are needed for what follows.

Let C be a complete irreducible algebraic curve defined over an algebraically closed field k. Throughout this paper we assume that C is a trigonal curve, in the sense that it is equipped with a g_3^1 but does not carry a g_2^1.

With the hypothesis that C is a Gorenstein curve of arithmetic genus $g \geq 5$, it was shown in [R-S] that there is an invariant of the trigonal curve C, which we call the Maroni invariant (cf. [M]). Moreover, the g_3^1 has a base point if and only if the Maroni invariant is zero (cf. [R-S, Theorem 2.1]). Since C is Gorenstein and non-hyperelliptic, we can embed it canonically in the projective space of dimension $g-1$ and we identify C with its image under the canonical embedding (cf. [H] or [Ro]). Let us assume that the Maroni invariant is zero. The canonical morphism of C is given by

$$(1 : x : x^2 : \cdots : x^{g-2} : y) : \tilde{C} \longrightarrow \mathbb{P}^{g-1}(k).$$

Moreover, with this choice of coordinate functions, C lies on the cone

$$S := \left\{ (x_0 : \cdots : x_{g-1}) \in \mathbb{P}^{g-1}(k) \;\middle|\; \operatorname{rank} \begin{pmatrix} x_0 & \cdots & x_{g-3} \\ x_1 & \cdots & x_{g-2} \end{pmatrix} < 2 \right\}.$$

Furthermore, the curve C passes through the vertex $V := (0 : \cdots : 0 : 1)$ of the cone. This point is singular and it is the base point of g_3^1.

The lines of S form a pencil $\{L_a |\ a \in k \cup \{\infty\}\}$. The non-singular locus $S \setminus \{V\}$ of the cone is described by the atlas consisting of the two charts

$$S \setminus L_\infty = \{(1 : a : \cdots : a^{g-2} : b) \mid (a, b) \in k^2\} \tilde{\to} \mathbb{A}^2(k)$$

and

$$S \setminus L_0 = \{(a^{g-2} : \cdots : a : 1 : b) \mid (a, b) \in k^2\} \tilde{\to} \mathbb{A}^2(k).$$

Associating to each irreducible curve in the affine plane the Zariski closure of its image under the local parameterization $\mathbb{A}^2(k) \tilde{\to} S \setminus L_\infty$, we obtain a bijective correspondence between the canonical curves on the cone S and the affine plane irreducible curves given by the equations

$$f(x, y) = c_2(x)y^2 + c_1(x)y + c_0(x) = 0 \tag{1}$$

such that $\deg(c_2(X)) \leq 2$, $\deg(c_1(X)) \leq g$, $\deg(c_0(X)) \leq 2g - 2$, and the equality holds at least once (cf. [R-S, Theorem 3.2]). Conversely, by intersecting canonical curves lying over the cone S with the pencil of lines $\{L_a \mid a \in k \cup \{\infty\}\}$, we see that they are trigonal; the g_3^1 has a base point and this point is the vertex of the cone (cf. [R-S, Theorem 3.6]).

The isomorphisms between two canonical curves on S are induced by automorphisms of $\mathbb{P}^{g-1}(k)$, leaving the cone invariant. Equivalently, these isomorphisms are given by

$$(x, y) \mapsto \left(\frac{\alpha_{11}x + \alpha_{12}}{\alpha_{21}x + \alpha_{22}}, \frac{\beta y + \beta_0 + \beta_1 x + \cdots + \beta_{g-2}x^{g-2}}{(\alpha_{21}x + \alpha_{22})^{g-2}} \right), \tag{2}$$

where $(\alpha_{ij}) \in GL_2(k)$, $\beta_i \in k$ for each $i \in \{0, 1, \ldots, g-2\}$, and $\beta \in k^*$ (cf. [R-S, Prop. 3.5]).

3. MODULI SPACES

Let C be a trigonal Gorenstein curve of arithmetic genus $g \geq 5$. We will always assume that the g_3^1 has a base point or, equivalently, the curve lies over the cone $S \subset \mathbb{P}^{g-1}$. Using transformations (2) we can normalize some of the coefficients of Eq. (1). Then, counting constants allows us to compute the dimension of the moduli space of such trigonal curves.

We divide the problem into three parts, according to the structure of the singularity at the base point of g_3^1, that is, the vertex of the cone S. A first case corresponds to the local degree of a generic divisor of g_3^1 at the base point V being equal to 2. In the other two cases, a generic divisor of g_3^1 has local degree 1 at V, but the base point may have either one or two tangent lines.

The points of the non-singular model $\tilde{C}$ will be viewed as branches of C, and correspond bijectively to the valuations of the field $k(C)$ of the rational functions on C. It follows that in our context $\tilde{C}$ is a hyperelliptic curve.

To study the curve C locally near the vertex V it suffices to analyze the affine curve

$$\{(x_0, x_1, \ldots, x_{g-2}) \in \mathbb{A}^{g-1}(k) \mid (x_0 : x_1 : \cdots : x_{g-2} : 1) \in C\}$$

near the origin. The coordinate functions of the affine curve are

$$\frac{x^0}{y}, \frac{x^1}{y}, \frac{x^2}{y}, \ldots, \frac{x^{g-2}}{y}.$$

A branch R of C is centered at V if and only if $\mathrm{ord}_R(x^j/y) > 0$ for each $j \in \{0, 1, 2, \ldots, g-2\}$ or, equivalently,

$$\mathrm{ord}_R(y) < 0 \quad \text{and} \quad \mathrm{ord}_R(y) < (g-2)\mathrm{ord}_R(x).$$

In this case the tangent line of the branch R centered at V is the line $L_{x(R)}$. We will consider $c_i(X)$ as a polynomial of formal degree $2g-2-i(g-2)$. If the degree of $c_i(X)$ is smaller than $2g-2-i(g-2)$, then will say that $c_i(X)$ has an infinite root of order $2g-2-i(g-2)-\deg c_i(X)$. By the method of the Newton polygon, the tangents of the branches of C centered at V are the lines L_a, where a varies over either the roots of $c_2(X) \neq 0$ or the roots of $c_1(X)$ if $c_2(X) = 0$.

Using transformations (2) we can assume that the line L_∞ is a tangent line of a branch centered at the vertex V. This means that $c_{22} = 0$ when $c_2(x) \neq 0$, and $c_{g1} = 0$ if $c_2(x) = 0$.

We will first assume that $c_2(x) = 0$. In this case, in general, the pencil of lines $\{L_a \mid a \in k \cup \{\infty\}\}$ cuts $C \setminus \{V\}$ transversely just at one point. Equivalently, the local degree of a generic divisor of g_3^1 at the base point V is equal to 2. Thus the non-singular model $\tilde{C}$ is rational, and each root of $c_1(X)$ of multiplicity n corresponds to a branch R centered at V with multiplicity n.

If $c_2(x) = 0$ and n is the multiplicity of the infinite root of $c_1(X)$, then $c_{g-n,1} \neq 0$. Now we normalize $c_{g-n,1} = c_{2g-2,0} = 1$ and $c_{g-n,0} = \cdots = c_{g-n+g-2,0} = 0$, so Eq. (1) reduces to

$$\text{(a)} \quad f(x,y) = (c_{01} + \cdots + x^{g-n})y + c_{00} + \cdots + c_{g-n-1,0}x^{g-n-1} + c_{2g-n-1,0}x^{2g-n-1} + \cdots + x^{2g-2} = 0.$$

By transforming $x \mapsto x + \alpha$, we are allowed to normalize $c_{2g-2-t,0} = 0$, where t is the largest power of the characteristic of the field k that divide $2g-2$. Notice that if the characteristic does not divide $2g-2$, we are normalizing $c_{2g-3,0} = 0$ and the only freedom left to us is to transform $(x, y) \mapsto (\alpha x, \alpha^{g-n-2} y)$ where $\alpha \in k^*$. We have proved the following result.

THEOREM 3.1. *For trigonal curves such that the local degree of a generic divisor of g_3^1 at the base point V is* 2 *and there is a branch centered at V of multiplicity n, the moduli space has dimension $2g-n-2$.*

The following results concern the case where $c_2(x) \neq 0$. In this case the pencil of lines $\{L_a \mid a \in k \cup \{\infty\}\}$ cuts $C \setminus \{V\}$ at two points, in general. Equivalently, a generic divisor of g_3^1 has local degree equal to 1 at V.

We deal first with the case where $c_2(X)$ has two different roots. One root we have been assuming as being infinity; the other we will assume to be equal to zero. Thus the point V is a node such that L_∞ and L_0 are the tangent lines of the branches centered at V. If we are working in characteristic different from 2, we can normalize $c_{11} = c_{21} = \cdots = c_{g-1,1} = 0$, and Eq. (1) reduces to

$$\text{(b)} \quad f(x, y) = xy^2 + (c_{01} + c_{g1} x^g) y + c_0(x) = 0.$$

In characteristic 2, we use a different normalization $c_{10} = c_{30} = \cdots = c_{2g-3,0} = 0$. The transformation that preserves the normalizations is $(x, y) \mapsto (\alpha x, \beta y)$, with α and $\beta \in k^*$. It follows that the dimension of the moduli space of such curves is equal to $2g-1$.

The only remaining case corresponds to $c_2(X)$ having a double root, at infinity. In characteristic different from 2, we can normalize $c_{01} = c_{11} = \cdots = c_{g-2,1} = 0$, and Eq. (1) reduces to

$$\text{(c)} \quad f(x, y) = y^2 + (c_{g-1,1} x^{g-1} + c_{g1} x^g) y + c_0(x) = 0.$$

And, in characteristic 2, we normalize $c_{00} = c_{20} = \cdots = c_{2g-4,0} = 0$. We are still allowed to transform $(x, y) \mapsto (\alpha x + \gamma, \beta y + \delta(x))$, where $\delta(x)$ depends only on the coefficients of $f(x, y)$, α, β, γ, and it is chosen to ensure that we are preserving normalizations. We conclude that the dimension of the moduli space of such curves is equal to $2g-2$.

THEOREM 3.2. *For trigonal curves such that the local degree of a generic divisor of g_3^1 at the base point V is* 2, *the moduli space is of degree $2g-1$, if V is a node, and it is of dimension $2g-2$, if there is only one tangent line of the branches centered at V.*

4. LOCAL PROPERTIES

As in previous sections C is a trigonal Gorenstein curve, carrying a g_3^1 with a base point. And we are realizing C as a canonical curve lying over the cone S. We want to describe local properties of the curve as singular degrees and, principally, hermitian invariants, which will be fundamental for the study of non-classical curves that we carry out in the next sections.

The discussion of the local properties of C near a point depends on whether the point coincides with the vertex or not. We begin by considering a point different from V, and we choose the chart $S \setminus L_\infty \tilde{\to} \mathbb{A}^2(k)$. A point

$$P = (1 : a : a^2 : \cdots : a^{g-2} : b) \in C \setminus L_\infty$$

is a singular point of C if a is a multiple root of the polynomial discriminant $\Delta(X) = c_1(X)^2 - 4c_2(X)c_0(X)$, in characteristic different from 2. Moreover, denoting by δ_P its local degree, we have

$$2\delta_P = \operatorname{ord}_a \Delta(x), \qquad \text{when the order is even,}$$

and

$$2\delta_P + 1 = \operatorname{ord}_a \Delta(x), \qquad \text{when the order is odd.}$$

The *hermitian invariants* of a branch of C with respect to the canonical morphism are its intersection multiplicities with hyperplanes of $\mathbb{P}^{g-1}(k)$. For each branch of C, there are exactly g hermitian invariants $j_0 < j_1 < \cdots < j_{g-1}$, where $j_0 = 0$ and j_1 is the multiplicity of the branch.

LEMMA 4.1. *Let us consider a branch of C not centered in the vertex.*

If the branch is non-singular and its tangent line does not pass through the vertex, the $g-1$ integers $0, 1, 2, \ldots, g-2$ are hermitian invariants.

If the branch is non-singular and its tangent line passes through the vertex, the hermitian invariants are $0, 1, 2, 4, \ldots, 2g-4$.

If the branch is singular, then its tangent line does not pass through the vertex and the $g-1$ integers $0, 2, 4, \ldots, 2g-4$ are the hermitian invariants.

Proof. We can assume that the branch is in the chart $S \setminus L_\infty \tilde{\to} \mathbb{A}^2(k)$. We denote by e its intersection multiplicity with the line of S passing through its center, and it follows that $e \leq 2$. The hermitian invariants are just the intersection multiplicities with the affine curves cut out by the equation

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{g-2} x^{g-2} + \beta y = 0,$$

where $(\alpha_0 : \alpha_1 : \cdots : \alpha_{g-2} : \beta) \in \mathbb{P}^{g-1}(k)$. By taking $\beta = 0$ or, equivalently, intersecting the branch with planes passing through the vertex V, we deduce that the $g-1$ integers $0e, 1e, 2e, \ldots, (g-2)e$ are invariants. ∎

Now, let us study the curve C locally near the point V. We use Hironaka's formula (cf. [Hi]) to calculate the singular degree δ_V of the vertex V. If $\deg_Y(f(X,Y)) = 1$, then the singular degree of the vertex is $\delta_V = g$. If $\deg_Y(f(X,Y)) = 2$, we have

$$2\delta_V - 2 = 2g - \deg(\Delta(X)) \qquad \text{if } \deg(\Delta(X)) \text{ is even,}$$

and

$$2\delta_V - 1 = 2g - \deg(\Delta(X)) \qquad \text{if } \deg(\Delta(X)) \text{ is odd.}$$

We recall that the local coordinate functions of C near V are x^i/y (i in $\{0,\dots,g-2\}$); a branch R is centered at V when $\mathrm{ord}_R(x^i/y) > 0$ for each i and, in this case, $L_{x(R)}$ is the tangent line of R. Thus these orders $\mathrm{ord}_R(x^i/y)$ are the hermitian invariants of the branch R centered at V. Their computation depends on whether $L_{x(R)}$ cuts C in another point different from V.

We first assume that $c_2(x) = 0$. Each tangent line of a branch centered at V cuts C only at V, so a branch of multiplicity n centered at V has the following sequence of hermitian invariants

$$(0, n, n+1, \dots, n+g-2) \tag{3}$$

and $n \geq \varepsilon_{g-1} - (g-2)$.

Now we treat the case $c_2(x) \neq 0$. As in the previous section, we assume that L_∞ is the tangent line of a branch centered at V. Let us suppose that $c_2(X)$ has two different roots. In this case the sequence of hermitian invariants of the branch with tangent line L_∞ is

$$\begin{aligned} &(0,1,2,\dots,g-1) && \text{if } c_{g1} \neq 0,\\ &(0,1,3,\dots,2g-3) && \text{if } c_{g1} = 0. \end{aligned} \tag{4}$$

Now we assume that $c_2(X)$ has a double root. If $c_{g1} \neq 0$, then V is a cusp. If $c_{g1} = 0$, then V is two-branched when $\deg(\Delta(X))$ is even and V is unibranched when $\deg(\Delta(X))$ is odd. The hermitian invariants of the branches centered at V are

$$\begin{aligned} &(0,2,3,\dots,g-1,g) && \text{if } c_{g1} \neq 0,\\ &(0,1,2,\dots,g-2,g-1) && \text{if } c_{g1} = 0 \text{ and } \deg(\Delta(X)) \text{ is even,}\\ &(0,2,4,\dots,2g-2) && \text{if } c_{g1} = 0 \text{ and } \deg(\Delta(X)) \text{ is odd.} \end{aligned} \tag{5}$$

5. NON-CLASSICAL TRIGONAL GORENSTEIN CURVES

In this section we establish some general necessary conditions for a curve to be non-classical, in terms of the types and configurations of their branches and singular points. In order to state these criteria, let us introduce some notation.

Let C be a canonical curve of arithmetic genus g. In [Sc], Schmidt proved that there is a sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{g-1})$ with respect to the canonical morphism, called the sequence of generic contact orders, such that $j_i(Q) = \varepsilon_i$ for all but finitely many branches Q of C, where $j_i(Q)$ denotes the hermitian invariant of Q (intersection multiplicity with a hyperplane of $\mathbb{P}^{g-1}(k)$). And it has been proved by Matzat [Ma] that $j_i(Q) \geq \varepsilon_i$ $(i = 0, 1, \ldots, g-1)$ for every branch Q of C. In fact, the sequence ε of the generic contact orders is the smallest one with respect to the lexicographic ordering, such that the *generalized wronskian determinant*

$$W_t^{\varepsilon}(z_0, \ldots, z_{g-1}) = \det(D_t^{(\varepsilon_i)} z_j)_{0 \leq i,\, j \leq g-1}$$

is not identically zero, where $D_t^{(\varepsilon_i)}$ denotes the ε_ith Hasse–Schmidt derivation.

A curve is said to be *non-classical* if the sequence $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{g-1})$ of the generic orders differs from the classical sequence $(0, 1, \ldots, g-1)$. This happens if and only if the *wronskian determinant*

$$W_t(z_0, \ldots, z_{g-1}) = \det(D_t^i z_j)_{0 \leq i,\, j \leq g-1}$$

is identically zero. When the characteristic of k is a prime number p, then the sequence ε satisfies the *p-adic criterion*: If $\delta \in \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{g-1}\}$, then each non-negative integer p-adically smaller than δ also belongs in $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{g-1}\}$. If $p = 0$ or $p > 2g-2$, then $\varepsilon = (0, 1, \ldots, g-1)$, and hence the curve is classical (cf. [S-V]). The hermitian invariants of each branch of a non-classical curve satisfy the *congruence criterion* (cf. [S-V, Corollary 1.9]):

$$\det\left(\binom{j_i(Q)}{n}\right)_{0 \leq i,\, n \leq g-1} \equiv 0 \pmod{p}.$$

Here we consider C a curve of arithmetic genus $g \geq 5$, trigonal, Gorenstein, and with Maroni invariant zero. In our setting the wronskian determinant is given by

$$W_x(1, x, x^2, \ldots, x^{g-2}, y) = D_x^{(g-1)} y.$$

Furthermore, we deduce from Lemma 4.1 that the sequence of generic orders is either of the type $(0, 1, 2, 3, \ldots, g-2, \varepsilon_{g-1})$, with $g - 1 \leq$

$\varepsilon_{g-1} \leq 2g-2$, or of the type $(0, 1, 2, 4, \ldots, 2g-4)$. The last sequence is possible only in characteristic 2. Since we are working with a non-classical curve, each integer p-adically smaller than the $(g-1)$th order of $(0, 1, 2, 3, \ldots, g-2, \varepsilon_{g-1})$ must be smaller than $g-1$. In particular, it is not possible that $p = g-1$.

The criterion of the wronskian determinant to decide whether a curve is non-classical is usually not practical to be checked through computations. That is why we also establish some necessary criteria, expressed in terms of the types and configurations of the branches, that are much easier to handle.

LEMMA 5.1. *If a branch of C admits the sequence* $(0, 2, 4, \ldots, 2g-2)$ *as hermitian invariants, then* $p = 2$.

If a branch of C admits the sequence $(0, n, n+1, \ldots, n+g-2)$ *as hermitian invariants, then p divides*

$$\binom{g+n-2}{n-1}.$$

In particular, if $n = 2$, *then p divides the arithmetic genus of C.*

If a branch of C admits the sequence $(0, 1, 3, \ldots, 2g-3)$, *then* $p = 2$ *or p divides*

$$\binom{2g-3}{g-1}.$$

Proof. Let P, Q, and R be three branches of C with the sequences of hermitian invariants $j(P) = (0, 2, 4, \ldots, 2g-2)$, $j(Q) = (0, n, n+1, \ldots, n+g-2)$, and $j(R) = (0, 1, 3, \ldots, 2g-3)$. Since C is non-classical, by the congruence criterion it is enough to see that

$$\det\left(\binom{j_i(P)}{s}\right) = 2^{g(g-1)/2},$$

$$\det\left(\binom{j_i(Q)}{s}\right) = \binom{g+n-2}{n-1},$$

and

$$\det\left(\binom{j_i(R)}{s}\right) = 2^{(g-2)(g-3)/2}\binom{2g-3}{g-1},$$

to obtain each of the three claims in the statement. ∎

Now we discuss briefly each of the three cases of non-classical curves corresponding to (a), (b), and (c) in characteristic different from 2. The characteristic 2 case will be discussed later.

Let us suppose that the trigonal curve C is given by (a). Since we are assuming that C is non-classical, the multiplicity of the branches centered at V is at least equal to 2 (cf. (3)) and therefore $c_{g-1,1} = c_{g1} = 0$.

We assume now that C is given by (b). In this case the sequence of hermitian invariants of the branch with tangent line L_∞ is $(0, 1, 3, \ldots, 2g-3)$ and hence $c_{g1} = 0$, since the classical sequence is excluded (cf. (4)). It follows from Lemma 5.1 that $\binom{2g-3}{g-1}$ is a multiple of the prime p.

Finally, we assume that C is given by (c). Combining Lemma 5.1 with (5) we see that $c_{g1} \neq 0$. In this case there exists only one branch of C centered at V. It follows from Lemma 5.1 that the characteristic of the constant field divides the arithmetic genus of C.

We observe that these necessary conditions we have been deriving are, in general, not sufficient to ensure that C is non-classical. On the other hand, they can be used to exclude the possibility of non-classical curves in some situations, as illustrated by the following proposition.

PROPOSITION 5.2. *If C is a non-classical trigonal curve of arithmetic genus $g \geq 5$ lying over a cone, then the characteristic of the constant field is different from $g-2$.*

Proof. Let us suppose that the characteristic of the constant field is $p = g-2$ and, nevertheless, C is non-classical. By the discussion above C is not given by (c), since p does not divide g.

If C is given by (b), then there is a branch centered in V which has the hermitian invariant sequence $(0, 1, 3, 5, \ldots, 2g-3)$. Hence (cf. Lemma 5.1) p divides

$$\binom{2g-3}{g-1} = \frac{(2g-3)(2g-4)!}{(g-2)!(g-2)!(g-1)},$$

which is not possible.

Let us suppose that C is given by (a) and let n be the multiplicity of a branch centered in V. In this case p divides

$$\binom{g+n-2}{n-1} = \frac{(g+n-2)!}{(n-1)!(g-2)!(g-1)}.$$

Thus $n = g-2$ and there is another branch centered in V which has multiplicity 2. It follows that p would divide g (cf. Lemma 5.1). ∎

6. SOME RELEVANT SPECIAL CASES

In this section we use the criteria obtained in Section 5 to study non-classical curves in some particular situations: $p = 2g-3$, $p = 2$, $p = g$. Combined with Proposition 5.2 and a direct analysis of the case $p = g = 5$, this leads to a complete classification of non-classical curves of genus 5, which we obtain in Section 7.

6.1. *Characteristic Equal to* $2g-3$

We now deal with the case where $2g-3$ is a prime number and it is equal to the characteristic of k. Recall that this is the largest value of the characteristic of the constant field compatible with C being non-classical.

THEOREM 6.1. *If the characteristic of the constant field is* $2g-3$, *then the sequence of orders of the canonical morphism of the non-classical curve* C *is* $(0, 1, 2, 3, \ldots, g-2, 2g-3)$, *the non-singular model* $\tilde{C}$ *is rational, and* C *is given by one of the following equations*

$$\text{(i)} \quad f(x, y) = xy^2 + (1 + x^{g-1})^2 = 0,$$

$$\text{(ii)} \quad f(x, y) = y + x^{2g-2} = 0.$$

If C *is given by* (i), *then there are no singular branches. If* C *is given by* (ii), *then there is only one singular branch, and this one is centered in the vertex* V.

Proof. The sequence of the generic orders is $\varepsilon = (0, 1, 2, 3, \ldots, g-2, \varepsilon_{g-1})$, where $\varepsilon_{g-1} = 2g-3$, by the p-adic criterion (cf. [S-V, Corollary 1.9]).

By Lemma 5.1, C does not admit (c). When C is given by (a) and n is the multiplicity of a root, finite or infinite, of the polynomial $c_1(X)$, then $n \geq \varepsilon_{g-1} - (g-2) = g-1$. So the equation becomes

$$f(x, y) = y + c_{g-1,0}x^{g-1} + \cdots + c_{2g-2,0}x^{2g-2} = 0.$$

Computing the wronskian determinant with respect to x we have

$$\begin{aligned} W_x(1, x, x^2, \ldots, x^{g-2}, y) = D_x^{g-1}y &= \frac{1}{(g-1)!}\frac{d^{g-1}y}{dx} \\ &= -\sum_{i=0}^{g-2} \binom{g-1+i}{g-1} c_{g-1+i,0}x^i, \end{aligned}$$

and, necessarily, $c_{g-1,0} = c_{g0} = \cdots = c_{2g-4,0} = 0$. Thus

$$f(x, y) = y + c_{2g-3,0}x^{2g-3} + c_{2g-2,0}x^{2g-2}.$$

Normalizing $c_{2g-2,0} = 1$ and transforming

$$(x, y) \mapsto (x - c_{2g-3,0}, y + c_{2g-3,0}^{2g-3}x),$$

we obtain $f(x, y) = y + x^{2g-2} = 0$. The unique singular point is V, so the unique singular branch is centered in V.

We assume now that C is given by (b). By Lemma 4.1 and by the p-adic criterion the possible sequences of hermitian invariants of the branches not centered in the vertex V are

$$(0, 1, 2, 3, \dots, g-2, 2g-3),$$
$$(0, 1, 2, 3, \dots, g-2, 2g-2),$$

and

$$(0, 2, 4, 6, \dots, 2g-4, 2g-3).$$

Thus the lines of the cone S are not tangent to any of these branches, since they never admit the sequence $(0, 1, 2, 4, 6, \dots, 2g-4)$. This means that if (a, b) is a point of the affine plane curves defined by $f(x, y) = 0$ such that $f_y(a, b) = 2ab = 0$, then $f_x(a, b) = 0$; that is, the point $(1 : a : a^2 : \cdots : a^{g-2} : 0)$ is a singular point of C. In this way we see that all roots of the polynomial $c_0(X) = c_{00} + c_{10}X + \cdots + c_{2g-2,0}X^{2g-2}$ are multiple. We claim that the multiplicity of each root is even and so, by blowing-up, the singular point $(1 : a : a^2 : \dots : a^{g-2} : 0)$ has two non-singular branches.

Indeed, let us suppose that the multiplicity of some root a is odd. In this case, by blowing-up, we see that there is only one branch centered in $(1 : a : a^2 : \cdots : a^{g-2} : 0)$; this branch is singular and hence has the hermitian invariants $(0, 2, 4, 6, \dots, 2g-4, 2g-3)$. By the local parameterization $\mathbb{A}^2(k) \tilde{\to} S \setminus L_\infty$, the plane line given by $y = 0$ corresponds to the curve, say E, on S defined by the intersection of S with the hyperplane

$$\{(x_0 : x_1 : \cdots : x_{g-1}) \in \mathbb{P}^{g-1}(k) \mid x_{g-1} = 0\}.$$

Thus the multiplicity of the root a of $c_0(X)$ is the intersection multiplicity of C with the non-singular curve E in the point $(1 : a : a^2 : \cdots : a^{g-2} : 0)$. This is the same as the intersection multiplicity of the branch centered in this point with the hyperplane given by $x_{g-1} = 0$, when such singular point is unibranched. It follows that this multiplicity, which we are supposing odd, is a hermitian invariant, and it is equal to $2g-3$. Since the degree of $c_0(X)$ is $2g-2$, the polynomial would have a simple root, which contradicts a previous conclusion. So, we have proved our claim.

Since $c_0(X)$ is a quadratic polynomial, we can write

$$c_0(X) = (d_0 + d_1X + \cdots + d_{g-2}X^{g-2} + d_{g-1}X^{g-1})^2,$$

where $d_0 = d_{g-1} = 1$. We see that $\tilde{C}$ is rational, because the projection of $\tilde{C}$ over E does not have ramified points. From this, it is possible to compute the wronskian determinant with respect to x by computing the $(g-1)$th derivative of $y = \alpha \sum_{i=0}^{g-1} d_i x^{i-1/2}$, where $\alpha \in k$ satisfies $\alpha^2 + 1 = 0$. Proceeding in this way, we find

$$\begin{aligned} D_x^{g-1} y &= \frac{1}{(g-1)!} \frac{d^{g-1}y}{dx} \\ &= \frac{1}{2^{g-1}(g-1)!} \sum_{i=0}^{g-1} \left(\prod_{j=0}^{g-2} (2(i-j)-1) \right) d_i \; x^{i-(2g-1)/2}. \end{aligned}$$

Observe that the factor $2(i-j)-1$ is congruent to zero mod $2g-3$ if and only if $i=0$ and $j=g-2$. It follows that $d_i=0$ for each $1\leq i\leq g-2$, and so the equation is indeed as in (i). ∎

The *weight* of a branch Q is defined by $\omega(Q):=\operatorname{ord}_Q(W_t^{\varepsilon}(x_0,\ldots,x_{g-1}))$, where t is a local parameter at Q and $\min\{\operatorname{ord}_Q(x_j)\}=0$. See [S-V]. It satisfies

$$\omega(Q)\geq\sum_{i=0}^{g-1}(j_i(Q)-\varepsilon_i)$$

and the equality holds if and only if

$$\det\left(\binom{j_i(Q)}{\varepsilon_n}\right)\not\equiv 0 \pmod{p}.$$

The finitely many branches $Q\in\tilde{C}$ whose weights are positive, or in other words, whose sequences $(j_0,j_1,\ldots j_{g-1})$ of hermitian invariants differ from the generic sequence ε, are called *Weierstrass branches*. The number of Weierstrass branches, counted according to their weights, is given by the formula

$$\sum_{Q\in\tilde{C}}\omega(Q)=\left(\sum\varepsilon_i\right)(2\tilde{g}-2)+g(2g-2).$$

As a scholium of the proof of Theorem 6.1, we obtain that the sequence of hermitian invariants of the Weierstrass branches not centered in V is

$$(0,1,2,3,\ldots,g-2,2g-2),$$

when the characteristic of k is $2g-3$. Furthermore, if C is given by the second equation of Theorem 6.1, then the branch centered in V has sequence $(0,g,g+1,\ldots,2g-3,2g-2)$. If C is given by the first equation of Theorem 6.1, then the two branches centered in V have sequence $(0,1,3,5,\ldots,2g-3)$ (cf. (4)). Therefore we have determined all sequences of hermitian invariants of the Weierstrass branches.

PROPOSITION 6.2. *If the characteristic of the constant field is* $2g-3$, *then for each Weierstrass branch* Q *of* C *we have*

$$\omega(Q)=\sum_{i=0}^{g-1}(j_i(Q)-\varepsilon_i).$$

Proof. If Q is a Weierstrass branch with the sequence $(0,1,2,3,\ldots,g-2,2g-2)$, then

$$\det\left(\binom{j_i(Q)}{\varepsilon_n}\right)=\binom{2g-2}{2g-3}=2g-2.$$

When $(0, 1, 3, 5, \dots, 2g-3)$ is the sequence of the branch Q, then we can write

$$\det\left(\begin{pmatrix} j_i(Q) \\ \varepsilon_n \end{pmatrix}\right) = \begin{pmatrix} 2g-3 \\ 2g-3 \end{pmatrix} \det\left(\begin{pmatrix} j_i(Q) \\ \varepsilon_n \end{pmatrix}\right),$$

where we consider $0 \le i \le g-1$ and $0 \le n \le g-1$ on the left-hand side, and $0 \le i \le g-2$ and $0 \le n \le g-2$ on the right-hand side. Moreover, the right-hand side is the same as

$$\prod_{i>s} \frac{j_i(Q) - j_s(Q)}{i-s},$$

since $\varepsilon_n = n$ for $0 \le n \le g-2$.

It remains to assume that the branch Q has the sequence of hermitian invariants equal to $(0, g, g+1, \dots, 2g-2)$. Then we have

$$\omega(Q) \ge \sum_{i=0}^{g-1} (j_i(Q) - \varepsilon_i) = (g-2)(g-1) + 1$$

and it is easy to see that the number on the right-hand side above is equal to the number of Weierstrass branches counted according to their weights minus one.

On the other hand, a branch Q as above can exist only when C is given by the second equation of Theorem 6.1. And, in this case, there exists another Weierstrass branch that is centered at $(1:0:\dots:0)$ and has weight one and $(0, 1, 2, 3, \dots, g-2, 2g-2)$ as the sequence of hermitian invariants. This means that the inequality above must be an equality, and the proof is finished also in this last case. ∎

6.2. *Characteristic Equal to* 2

By Proposition 4.1, when the characteristic of the constant field is 2, the two possible sequences of the canonical morphism are

$$(0, 1, 2, 3, \dots, g-2, \varepsilon_{g-1}) \qquad \text{and} \qquad (0, 1, 2, 4, \dots, 2g-4).$$

Moreover, the second case occurs if and only if all tangent lines of the branches pass through the vertex of the cone, and in this case C is given by $f(x, y) = c_2(x)y^2 + c_0(x)$.

THEOREM 6.3. *If the characteristic of the constant field is equal to two and the sequence of orders of the trigonal curve C is $(0, 1, 2, 4, \dots, 2g-4)$, then C is given by one of the following equations*

$$\text{(i)} \quad f(x, y) = xy^2 + (1 + \lambda_1 x + \lambda_2 x^2 + \dots + x^{g-1})^2 = 0,$$

$$\text{(ii)} \quad f(x, y) = y^2 + c_{10}x + \dots + c_{i-4,0}x^{i-4} + x^i + x^{2g-2} = 0,$$

where $1 \le i \le 2g-3$ *and* $c_{i-4,\,0} = 0$ *if* $i - 4 < 0$.

Proof. Let us suppose that the sequence is $(0, 1, 2, 4, \dots, 2g-4)$. This is enough to ensure that C is non-classical. In this case C is given by one of the equations

$$xy^2 + c_{00} + \cdots + c_{2g-2,0}x^{2g-2} = 0,$$
$$y^2 + c_{00} + \cdots + c_{2g-2,0}x^{2g-2} = 0.$$

If C is given by the first equation then we normalize $c_{00} = c_{2g-2,\,0} = 1$ and $c_{10} = c_{30} = \cdots = c_{2g-3,\,0} = 0$. Thus we obtain the equation

$$f(x, y) = xy^2 + (1 + \lambda_1 x + \lambda_2 x^2 + \cdots + x^{g-1})^2 = 0,$$

where the coefficients are uniquely determined up to the transformation that changes the branches centered in V. When $y^2 + c_{00} + \cdots + c_{2g-2,\,0}x^{2g-2} = 0$ gives C, we normalize $c_{2g-2,0} = 1$, and $c_{00} = c_{20} = \cdots = c_{2g-4,\,0} = 0$, to obtain $y^2 + c_{10}x + c_{30}x^3 + \cdots + c_{2g-3,\,0}x^{2g-3} + x^{2g-2} = 0$. By the irreducibility of the last equation, at least one of the coefficients is not zero, so we are allowed to normalize two more coefficients. Thus we have the equation

$$f(x, y) = y^2 + c_{10}x + \cdots + c_{i-4,\,0}x^{i-4} + x^i + x^{2g-2} = 0,$$

where $1 \le i \le 2g-3$ and $c_{i-4,\,0} = 0$ if $i - 4 < 0$. ∎

Each equation of the theorem above gives a *strange* curve since the tangent lines of the curve pass through the vertex V. The non-singular model $\tilde{C}$ of C is a hyperelliptic curve of *inseparable type*.

Now let us suppose that the sequence of orders is $(0, 1, 2, 3, \dots, g-2, \varepsilon_{g-1})$. In general, it is very hard to determine conditions over the coefficients of

$$f(x, y) = c_2(x)y^2 + c_1(x)y + c_0(x)$$

to ensure that C is non-classical, because for this one must impose the vanishing of the coefficients of the wronskian determinant modulo $f(x, y)$. Here (having in mind the case $g = 5$), we will assume that $g = 2^n + 1$, where $n \ge 2$. This hypothesis gives us some geometrical properties that minimize the computations. In this way we can classify all non-classical trigonal curves of genus $g = 2^n + 1$ in characteristic 2.

Under this hypothesis, the last order ε_{g-1} must be 2-adically bigger than $g - 1 = 2^n$, so $\varepsilon_{g-1} = 2^{n+1} = 2g - 2$ and the sequence of generic contact orders is

$$(0, 1, 2, 3, \dots, g-2, 2g-2).$$

This means that, in each branch of C, the osculator hyperplane cuts C with multiplicity equal to the degree of C. It follows that every point is unibranched. In particular, if $c_2(X) \neq 0$ then it must have a double root. We consider two cases corresponding to $c_2(x) = 0$ and $c_2(x) \neq 0$, respectively. In the second case, $c_2(x) = 1$, since we are assuming that the infinity is a root.

If C is given by $c_1(x)y + c_0(x) = 0$, the multiplicity of every root of $c_1(X)$ cannot be smaller than $\varepsilon_{g-1} - (g-2) = g$. Thus we have the equation

$$f(x, y) = y + c_{g-1,\,0}x^{g-1} + \cdots + x^{2g-2} = 0.$$

The intersection multiplicity, in $(0, 0)$, of the plane curve given by $f(x, y) = 0$ with the plane line given by $y = 0$ is a hermitian invariant, since it is the intersection multiplicity of C with the hyperplane

$$\{(x_0 : x_1 : \cdots : x_{g-1}) \in \mathbb{P}^{g-1}(k) | x_{g-1} = 0\}$$

in the branch centered in $(1 : 0 : 0 : \cdots : 0)$. Thus this multiplicity must be equal to $2g - 2$. Therefore, in this case, C is given by $f(x, y) = y + x^{2g-2} = 0$.

Let us suppose C is given by $f(x, y) = y^2 + c_1(x)y + c_0(x)$. The assumption on the sequence ensures that $c_1(x) = c_{01} + c_{11}x + \cdots + c_{g1}x^g$ is not identically zero. Then, by transforming $x \mapsto x + \alpha$, we can assume that $c_{01} \neq 0$. The sequence of hermitian invariants of the branch centered in V is $(0, 2, 4, \ldots, 2g-4, 2g-2)$. Thus $c_{g1} = 0$ (cf. (5)) and we are allowed to normalize $c_{2g-2,\,0} = 1$. We can replace y by $y + \beta_0 + \beta_1 x + \cdots + \beta_{g-2}x^{g-2}$, such that

$$f(x, y) = y^2 + (c_{01} + \cdots + c_{g-1,\,1}x^{g-1})y + c_{g-1,\,0}x^{g-1} + \cdots + x^{2g-2} = 0.$$

Since $f_y(0, 0) = c_{01} \neq 0$, the point $(1 : 0 : \cdots : 0)$ is non-singular and has the sequence $(0, 1, 2, \ldots, 2g-2)$ as the sequence of hermitian invariants. Since the intersection multiplicity of C with the hyperplane given by $x_{g-1} = 0$, in the branch centered in $(1 : 0 : \cdots : 0)$, is $2g - 2$, it follows that $c_{g-1,\,0} = \cdots = c_{2g-3,\,0} = 0$, and hence $f(x, y) = y^2 + c_1(x) + x^{2g-2} = 0$.

Moreover, there is no branch with tangent line passing through the vertex, since the sequence $(0, 1, 2, 4, \ldots, 2g-4)$ is not admitted. Thus, if the polynomial $c_1(X)$ has a root, it must be multiple and it must correspond to a singular point of C.

We claim that $c_1(X)$ does not have roots. To see this, let us suppose that there is a root, say a, and let R be a singular point corresponding to it. Then R must be unibranched. Indeed, if R were two-branched, then each branch would have the sequence of hermitian invariants equal to $(0, 1, 2, \ldots, g-2, 2g-2)$. Then there would exist a hyperplane cutting C with intersection multiplicity bigger than $2g - 2$, which is not possible.

Note that the multiplicity of the root a is a hermitian invariant of the only branch centered at R, so it must be even since the sequence of the branch is $(0, 2, 4, \ldots, 2g-2)$. But then, by blowing-up, we get that R is two-branched. We have reached a contradiction, so we may conclude that $c_1(x)$ is constant, and hence $f_x = 0$.

We have proved the following theorem.

THEOREM 6.4. *If the characteristic of the constant field is equal to* 2, $g = 2^n + 1$, *and the sequence of orders of the trigonal curve* C *is not* $(0, 1, 2, 4, \ldots, 2g-4)$, *then the sequence must be* $(0, 1, 2, 3, \ldots, g-2, 2g-2)$ *and* C *is given by one of the following equations*

(i) $f(x, y) = y + x^{2g-2} = 0$,

(ii) $f(x, y) = y^2 + y + x^{2g-2} = 0$.

6.3. *Genus Equal to the Characteristic*

THEOREM 6.5. *If the characteristic of the constant field is equal to the arithmetic genus of the non-classical curve* C, *then the sequence of orders of the canonical morphism is* $(0, 1, 2, \ldots, g-2, p)$. *If* C *is given by an equation of degree* 2 *with respect to* y, *then this equation is one of the following.*

(i) $xy^2 + 1 + c_{10}x + \cdots + x^{2g-2} = 0$

such that

$$\frac{d^{p-1}x^{-1}\Delta(x)^{(p+1)/2}}{dx^{p-1}} = 0,$$

(ii) $y^2 + x^g y + c_0(x) = 0$,

where the polynomial discriminant $\Delta(X)$ *is not quadratic and*

$$\frac{d^{p-1}\Delta(x)^{(p+1)/2}}{dx^{p-1}} = 0.$$

Proof. We know that the sequence of the canonical morphism is of the type $(0, 1, 2, \ldots, g-2, \varepsilon_{g-1})$. Since $p = g$, it follows by the p-adic criterion (cf. [S-V, Corollary 1.9]) that each integer p-adically smaller than ε_{g-1} is smaller than $g-1$, and thus $\varepsilon_{g-1} = p$.

Since the wronskian determinant $W_x(1, x, x^2, \ldots, x^{g-2}, y)$ is identically zero, we must have $d^{p-1}y/dx^{p-1} = 0$, or, equivalently, the image of the differential $y\,dx$ with respect to the *Cartier operator* for $f(x, y) = 0$ must be zero (cf. [S-V2, Theorem 1.1]). So we have

$$\frac{\partial^{2p-2}(yf_y(x, y)f(x, y)^{p-1})}{\partial x^{p-1}\partial y^{p-1}} = 0.$$

The polynomial expression $yf_y(x,y)f(x,y)^{p-1}$ is of degree $2p$ with respect to y. Thus

$$\frac{\partial^{2p-2}(yf_y(x,y)f(x,y)^{p-1})}{\partial x^{p-1}\partial y^{p-1}} = -c_1(x)y^p - d(x),$$

where $d(x)$ is the coefficient of the expression $yf_y(x,y)f(x,y)^{p-1}$ seen as an element of $k[x][y]$.

If C is described by (b), then $yf_y(x,y)f(x,y)^{p-1} = 2xy^2(xy^2+c_0(x))^{p-1}$, and hence

$$\begin{aligned}\frac{\partial^{2p-2}(yf_y(x,y)f(x,y)^{p-1})}{\partial x^{p-1}\partial y^{p-1}} &= \frac{d^{p-1}(x^{(p-1)/2}c_0(x)^{(p-1)/2})}{dx^{p-1}}\\ &= \frac{d^{p-1}(x^{-1}\Delta(x)^{(p+1)/2})}{dx^{p-1}} = 0.\end{aligned}$$

Suppose that C is given by equation (c). In this case, $c_1(x) = c_{g-1,1}x^{g-1} + x^g$ and thus $c_{g-1,1} = 0$, since

$$\frac{\partial^{p-1}(c_1(x)y^p + d(x))}{\partial x^{p-1}} = (g-1)c_{g-1,1}y^p + \frac{d^{p-1}d(x)}{dx^{p-1}} = 0.$$

Defining $z := y + x^p/2$, we see that $dy/dx = dz/dx$. So the image of the differential zdx with respect to the Cartier operator for the equation $z^2 - \Delta(x) = 0$, where $\Delta(x) = x^{2p} - 4c_0(x)$, must be zero, too. Using the formula for the Cartier operator (cf. [S-V2, Theorem 1.1]) we finally get

$$\frac{\partial^{2p-2}(2z^2(z^2-\Delta(x))^{p-1})}{\partial x^{p-1}\partial z^{p-1}} = \frac{d^{p-1}\Delta(x)^{(p+1)/2}}{dx^{p-1}} = 0.$$

∎

Let us close this section with some remarks concerning the case when $g \le p$. We know that, by the p-adic criterion (cf. [S-V, Corollary 1.9]), $\varepsilon = (0, 1, 2, \ldots, g-2, p)$ is the sequence of orders of the canonical morphism of C. The sequence ε is the minimal one such that the generalized wronskian determinant

$$W_x^{\varepsilon}(1, x, x^2, \ldots, x^{g-2}, y) = D_x^{(p)}y$$

is not zero, so we must have

$$\frac{d^{g-1}y}{dx^{g-1}} = \frac{d^g y}{dx^g} = \cdots = \frac{d^{p-1}y}{dx^{p-1}} = 0.$$

When C is given by (b), the necessary condition (but not sufficient if $g < p$) $d^{p-1}y/dx^{p-1} = 0$ for C to be non-classical is equivalent to

$$\frac{d^{p-1}x^{-1}\Delta(x)^{(p+1)/2}}{dx^{p-1}} = 0.$$

Note also that, by Lemma 5.1, if C is given by (c) then p divides g. So, since we are supposing $g \le p$, we must have $p = g$.

7. NON-CLASSICAL TRIGONAL GORENSTEIN CURVES OF GENUS FIVE

In this section we give an application of the general results deduced from the previous ones, yielding a complete classification of arithmetic genus five non-classical trigonal Gorenstein curves with the Maroni invariant equal to zero.

Let C be such a curve. We are assuming that C is non-classical, and thus the characteristic of the constant field must be $p \in \{2, 3, 5, 7\}$. As we have seen in Proposition 5.2, it is not possible to have $p = g - 2 = 3$. First, we classify the cases $p = 7$ and $p = 2$, by applying the criteria we obtained in Section 6. The case $p = 5$ requires a more lengthy analysis.

If $p = 2g - 3 = 7$, then C is given by one of the following equations (cf. Theorem 6.1)

$$f(x, y) = xy^2 + (1 + x^4)^2 = 0,$$
$$f(x, y) = y + x^8 = 0.$$

Let us suppose that $p = 2$. Since $g = 5 = 2^2 + 1$, it follows from Theorems 6.3 and 6.4 that C is given by one of the following equations

$$f(x, y) = xy^2 + (1 + \lambda x + \mu x^2 + \gamma x^3 + x^4)^2 = 0,$$
$$f(x, y) = y^2 + \lambda x + \mu x^3 + x^7 + x^8 = 0,$$
$$f(x, y) = y^2 + \lambda x + x^5 + x^8 = 0,$$
$$f(x, y) = y^2 + x^3 + x^8 = 0,$$
$$f(x, y) = y^2 + x + x^8 = 0,$$
$$f(x, y) = y^2 + y + x^8 = 0,$$
$$f(x, y) = y + x^8 = 0.$$

From now on we deal with the case $g = p = 5$. Then the sequence of generic contact orders is $(0, 1, 2, 3, 5)$.

Let us first suppose that C is given by (a). Either the vertex is unibranched or there are two branches centered in it, one of multiplicity equal to 3 and the other of multiplicity equal to 2. By normalizing coefficients and computing the wronskian determinant with respect to x, we get that C is given by one of the following equations

$$f(x, y) = y + \lambda x^5 + \mu x^6 + x^8 = 0,$$
$$f(x, y) = x^2 y + 1 + \lambda x^7 + x^8 = 0.$$

We assume now that C is given by (b). By Theorem 6.5, the coefficients of this equation must satisfy the condition

$$\frac{d^4 x^{-1}\Delta(x)^6}{dx^4} = 0,$$

which gives five polynomial equations among the nine coefficients. Since we are allowed to normalize two coefficients, we conclude that the space of moduli has dimension equal to 2. Moreover, when the curve is given by (c), we can use Theorem 6.5 in the same way, to conclude that the space of moduli is also of dimension 2.

In each of these two cases above, the five polynomial equations define an affine variety. We illustrate the study of the geometry of these moduli varieties; in the generic situation, we consider the case when C is given by (b).

First we normalize $c_{80} = 1$. Since we are interested in the generic situation, we assume that $c_{70} \neq 0$ and normalize $c_{70} = 1$. Then we eliminate c_{20} and c_{60} from the five equations, thus getting a new set of three equations.

Now we substitute $c_{10} = \lambda c_{00}$ and $c_{30} = c_{00}(\delta - \lambda^3 c_{50})$ in the equations, because λ and δ defined in this way are invariants under the transformation that interchanges the two branches centered at the vertex V. Thus one of the equations becomes linear with respect to c_{40}, and another one becomes linear with respect to c_{00}. So we eliminate these coefficients, obtaining a single equation on the variables δ, λ and c_{50}.

Another invariant of the transformation that interchanges the branches centered at V is $\gamma = c_{50}^2 - (\delta/\lambda^3)c_{50}$, and so we substitute $c_{50}^2 = \gamma + (\delta/\lambda^3)c_{50}$. The equation in $k[\lambda, \delta, \gamma]$ that we find in this way is reducible and, in the generic case, has two factors. Fixing $\lambda \in k$ we treat the two factors of this polynomial equation as equations of affine plane curves. In one of them we substitute $Z = \delta$ and $W = \lambda^4\gamma$, and in the other $Z = \delta$ and $W = \lambda^3\gamma$.

In this way we obtain equations of a conic

$$\begin{aligned} &\lambda^2 - 2\lambda^3 + 2\lambda W - \lambda^2 W + \lambda^3 W - \lambda^4 W + W^2 \\ &\qquad -2\lambda Z + 2\lambda^2 Z + \lambda^3 Z - 2WZ - \lambda WZ + Z^2 = 0, \end{aligned} \tag{6}$$

and of a cubic

$$\begin{aligned} &1 + \lambda - 2\lambda^4 + \lambda^5 - 2W + \lambda^2 W - \lambda^3 W - \lambda^4 W + \lambda^5 W + \lambda^6 W - 2W^2 \\ &\quad - \lambda W^2 - \lambda^2 W^2 - 2\lambda^3 W^2 + W^3 + Z - \lambda Z - 2\lambda^2 Z + \lambda^3 Z - 2\lambda^4 Z - 2WZ \\ &\quad - \lambda WZ + \lambda^2 WZ + 2\lambda^3 WZ - 2W^2 Z - Z^2 + 2\lambda Z^2 + WZ^2 = 0. \end{aligned} \tag{7}$$

Note that the conic given by Eq. (6) has a singular point, which is $(-\lambda(\lambda^2 + 2\lambda + 4), -\lambda^2)$, so it is reducible. The cubic given by (7) has a singular infinity point, which is $(1:1:0)$. So we may project the cubic from this singular point over the line given by $W = 0$. From these two remarks we conclude that each of these equations determines δ as a function of λ and γ, and so these two last variables parameterize the moduli variety.

A similar analysis of the moduli variety can be carried out when the curve is given by (c). The calculations are longer, but follow along the same lines.

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